جامعة بيرزيت
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Control Systems

Introduction, and Modeling in Frequency Domain

Control system definition

A control system consists of *subsystems* and *processes* (or *plants)* assembled for the purpose of obtaining a desired *output* with desired *performance*, given a specified *input*. Figure 1.1 shows a control system in its simplest form, where the input represents a desired output.

FIGURE 1.1 Simplified description of a control system

For example, consider an elevator.

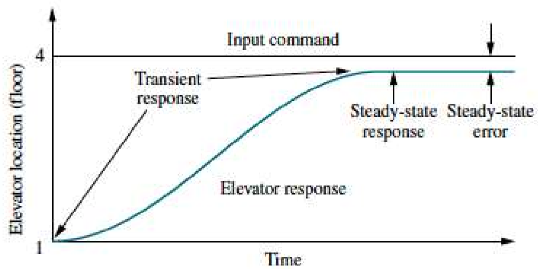
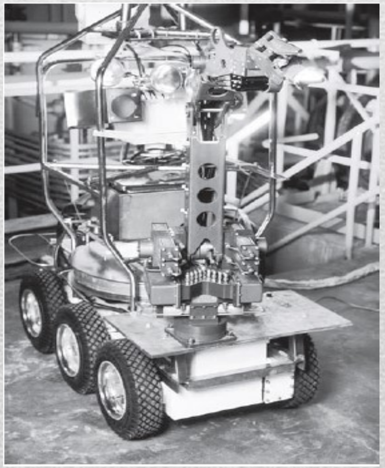


FIGURE 1.2 Elevator response

Advantages of control systems

We build control systems for four primary reasons:

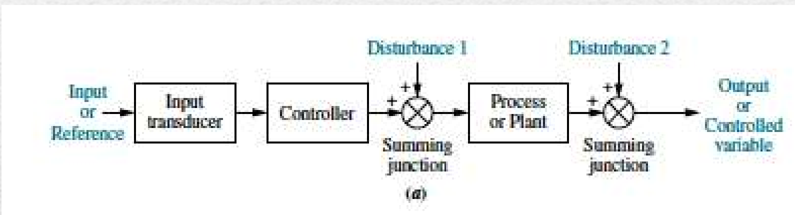
1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation for disturbances



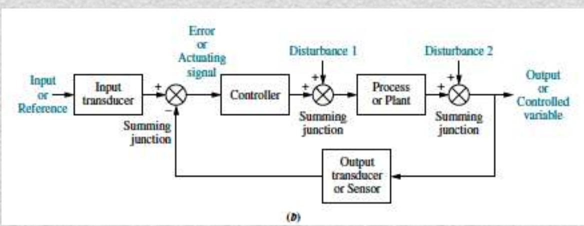
System configurations

In this section, we discuss two major configurations of control systems: open loop and closed loop.

Open-Loop Systems



Closed-Loop (Feedback Control) Systems



The design process

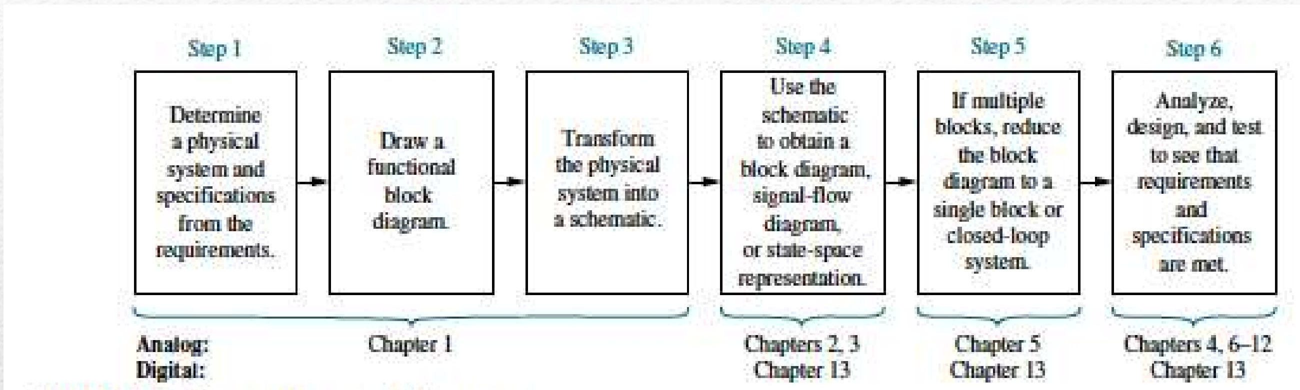
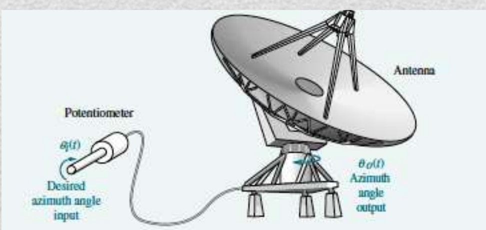
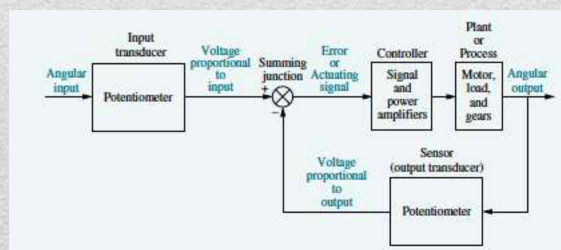
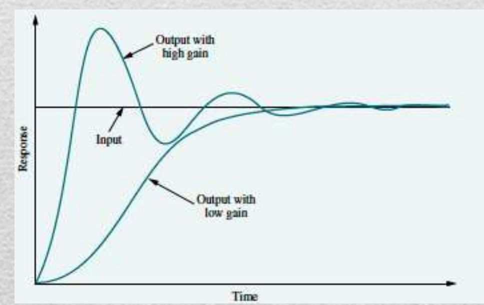


FIGURE 1.11 The control system design process

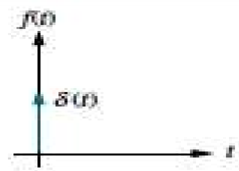
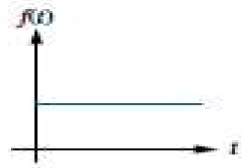
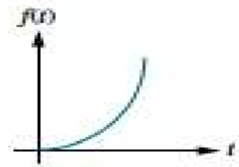
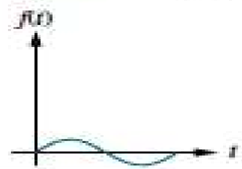






The design process

TABLE 1.1 Test waveforms used in control systems

Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

Modeling in the frequency domain

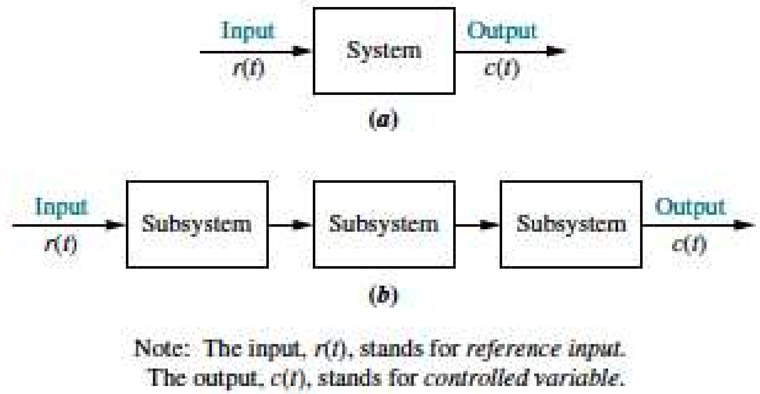
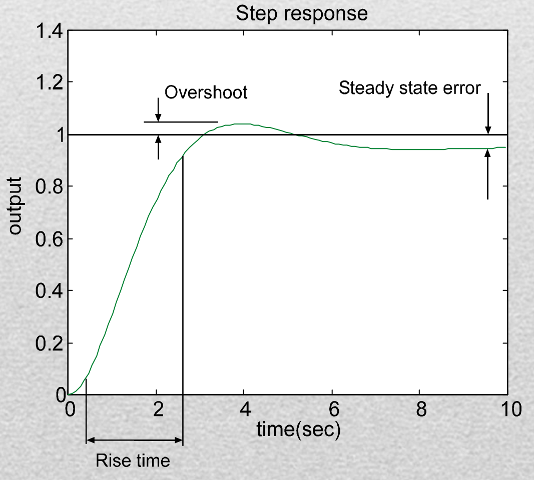


FIGURE 2.1 a. Block diagram representation of a system; b. block diagram representation of an interconnection of subsystems

- To understand system performance, a mathematical model of the plant is required
- This will eventually allow us to design control systems to achieve a particular specification



Laplace transform review

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

or

$$F(s) = \text{Laplace transform of } f(t) = \mathcal{L}[f(t)]$$

The defining equation above is also known as the **one-sided Laplace transform,** as the integration is evaluated from $t=0$ to ∞.

Laplace table

Item no.	f(t)	F(s)
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace table

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \quad (2.14)$$

SOLUTION: Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-) = 0$ and $\dot{y}(0-) = 0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \quad (2.15)$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} \quad (2.16)$$

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} \quad (2.17)$$

where, from Eq. (2.13),

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1 \quad (2.18a)$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2 \quad (2.18b)$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1 \quad (2.18c)$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \quad (2.19)$$

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

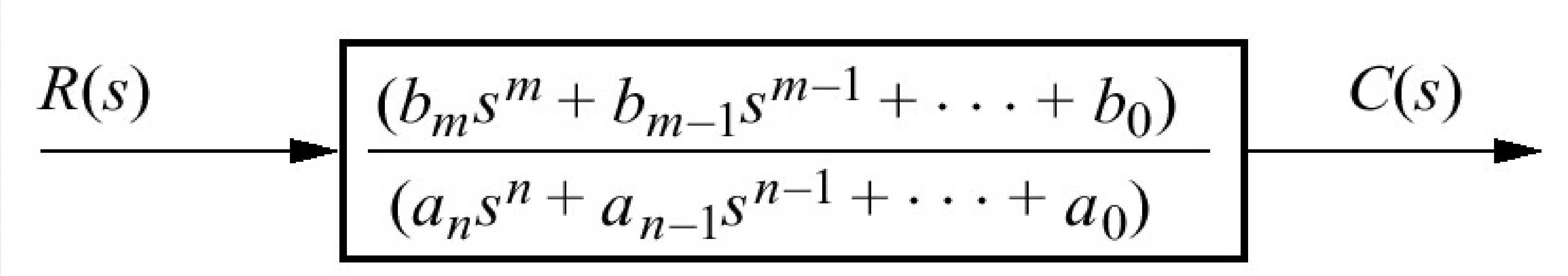
$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \quad (2.20)$$

TRANSFER FUNCTION

T.F of LTI system is defined as the Laplace transform of the impulse response, with all the initial condition set to zero

Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \quad (2.50)$$



Laplace table

Example 2.4

Transfer Function for a Differential Equation

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \tag{2.55}$$

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s) \tag{2.56}$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \tag{2.57}$$

Laplace table

Example 2.5

TryIt 2.6

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.60).

```
syms s
C=1/(s*(s+2))
C=ilaplace(C)
```

TryIt 2.7

Use the following MATLAB statements to plot Eq. (2.60) for t from 0 to 1 s at intervals of 0.01 s.

```
t=0:0.01:1;
plot...
 (t,(1/2-1/2*exp(-2*t)))
```

System Response from the Transfer Function

PROBLEM: Use the result of Example 2.4 to find the response, $c(t)$ to an input, $r(t) = u(t)$, a unit step, assuming zero initial conditions.

SOLUTION: To solve the problem, we use Eq. (2.54), where $G(s) = 1/(s+2)$ as found in Example 2.4. Since $r(t) = u(t)$, $R(s) = 1/s$, from Table 2.1. Since the initial conditions are zero,

$$C(s) = R(s)G(s) = \frac{1}{s(s+2)} \tag{2.58}$$

Expanding by partial fractions, we get

$$C(s) = \frac{1/2}{s} - \frac{1/2}{s+2} \tag{2.59}$$

Finally, taking the inverse Laplace transform of each term yields

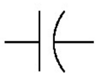
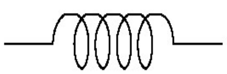
$$c(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} \tag{2.60}$$

ELECTRICAL NETWORK TRANSFER FUNCTION

- In this section, we formally apply the transfer function to the mathematical modeling of electric circuits including passive networks
- Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.“
- We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are Kirchhoff s laws.

ELECTRICAL NETWORK TRANSFER FUNCTION

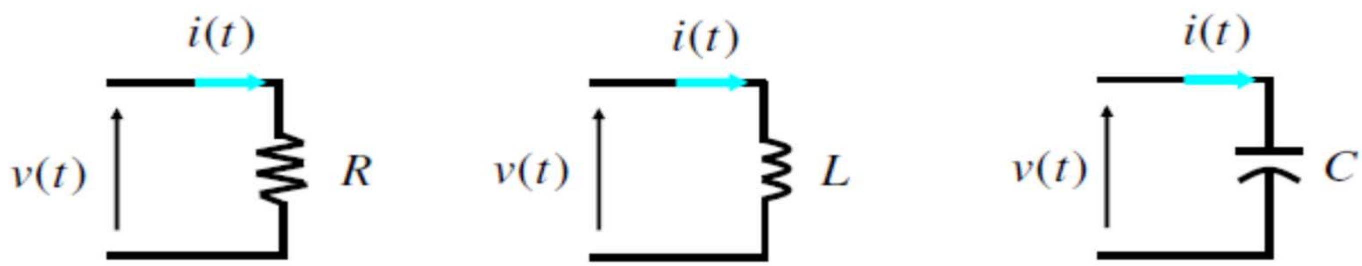
Table 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ = V (volts), $i(t)$ = A (amps), $q(t)$ = Q (coulombs), C = F (farads), R = Ω (ohms), G = ℧ (mhos), L = H (henries).

Modeling electrical element

Resistance | Inductance | Capacitance



$$v(t) = R\,i(t) \qquad v(t) = L\frac{di(t)}{dt} \qquad v(t) = v(0) + \frac{1}{C}\int_0^t i(t)dt$$

Laplace transform $\qquad (i(0)=0) \qquad (v(0)=0)$

$$\frac{V(s)}{I(s)} = R \qquad \frac{V(s)}{I(s)} = sL \qquad \frac{V(s)}{I(s)} = \frac{1}{sC}$$

Modeling – Kirchhoff's Voltage & Current Laws

Example 2.6

Transfer Function—Single Loop via the Differential Equation

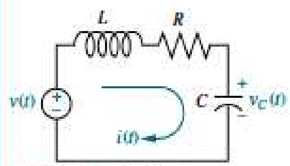


FIGURE 2.3 *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION: In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \qquad (2.61)$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \qquad (2.62)$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \qquad (2.63)$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \qquad (2.64)$$

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \qquad (2.65)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad (2.66)$$

as shown in Figure 2.4.

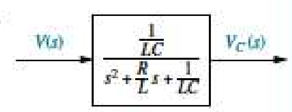


FIGURE 2.4 Block diagram of series *RLC* electrical network

Modeling – Kirchhoff's Voltage & Current Laws

Example 2.14

Transfer Function—Inverting Operational Amplifier Circuit

PROBLEM: Find the transfer function, $V_o(s)/V_i(s)$, for the circuit given in Figure 2.11.

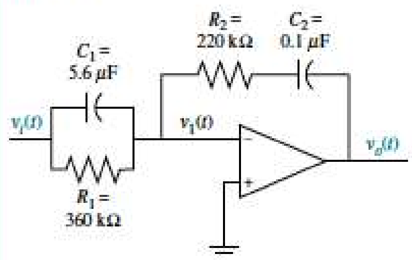


FIGURE 2.11 Inverting operational amplifier circuit for Example 2.14

SOLUTION: The transfer function of the operational amplifier circuit is given by Eq. (2.97). Since the admittances of parallel components add, $Z_1(s)$ is the reciprocal of the sum of the admittances, or

$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}} = \frac{360 \times 10^3}{2.016s + 1} \quad (2.98)$$

For $Z_2(s)$ the impedances add, or

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \quad (2.99)$$

Substituting Eqs. (2.98) and (2.99) into Eq. (2.97) and simplifying, we get

$$\frac{V_o(s)}{V_i(s)} = -1.232 \frac{s^2 + 45.95s + 22.55}{s} \quad (2.100)$$

The resulting circuit is called a PID controller and can be used to improve the performance of a control system. We explore this possibility further in Chapter 9.

Modeling – Summary (Electrical System)

- Modeling
  - Modeling is an important task!
  - Mathematical model
  - Transfer function
  - Modeling of electrical systems
- Next, modeling of mechanical systems

Translational Mechanical System T.F

- The motion of Mechanical elements can be described in various dimensions as translational, rotational, or combinations of both.
- Mechanical systems, like electrical systems have three passive linear components.
- Two of them, the spring and the mass, are energy-storage elements; one of them, the viscous damper, dissipate energy.
- The motion of translation is defined as a motion that takes place along a straight or curved path. The variables that are used to describe translational motion are **acceleration, velocity,** and **displacement.**

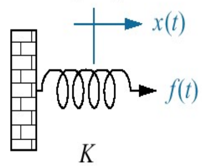
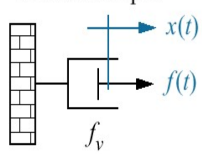
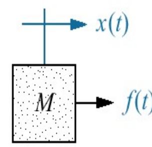
Translational Mechanical System T.F

Newton's law of motion states that the algebraic sum of external forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. The law can be expressed as

$$\sum Forces = Ma$$

Translational Mechanical System T.F

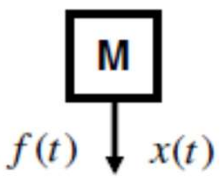
Table 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Modeling-mechanical elements

Mass

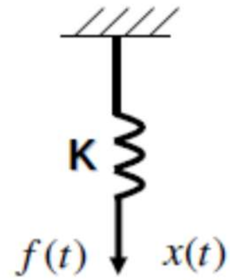


$$f(t) = M\ \ddot{x}(t)$$

$$x(0) = 0$$
$$\dot{x}(0) = 0$$

$$F(s) = Ms^2 X(s)$$

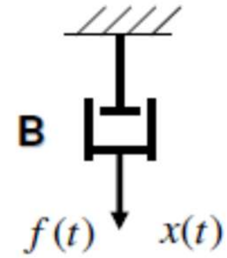
Spring



$$f(t) = K\ x(t)$$

$$F(s) = KX(s)$$

Damper

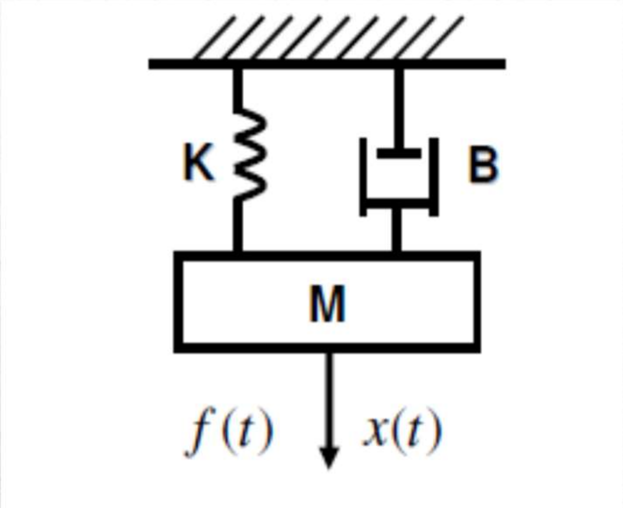


$$f(t) = B\ \dot{x}(t)$$

$$x(0) = 0$$

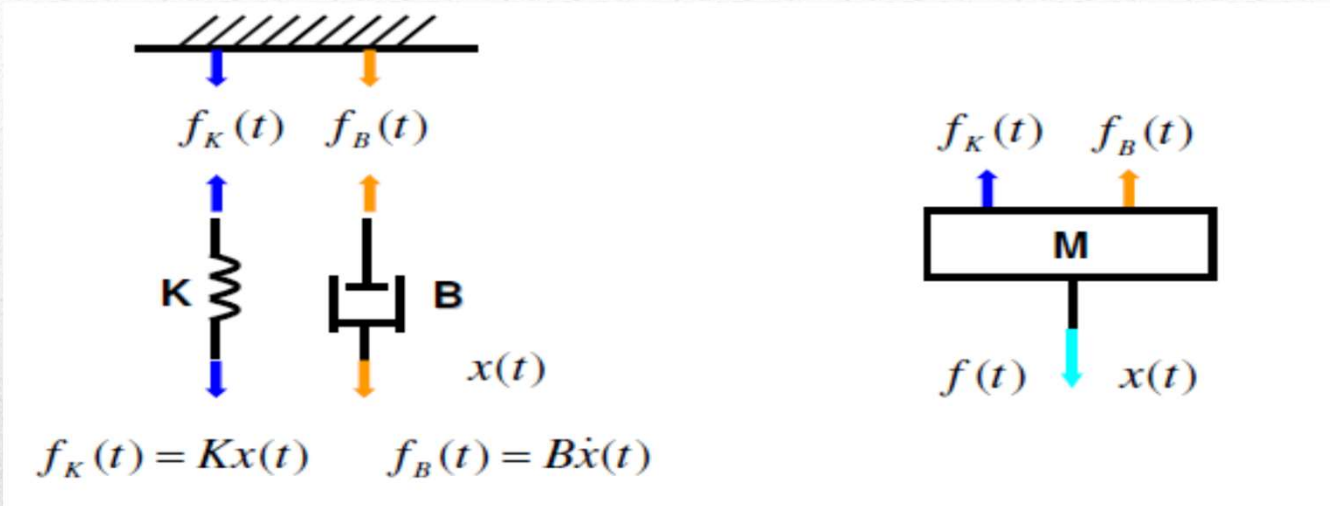
$$F(s) = BsX(s)$$

Modeling – Spring-Mass-Damper Systems



$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)$$

Modeling – Free Body Diagram



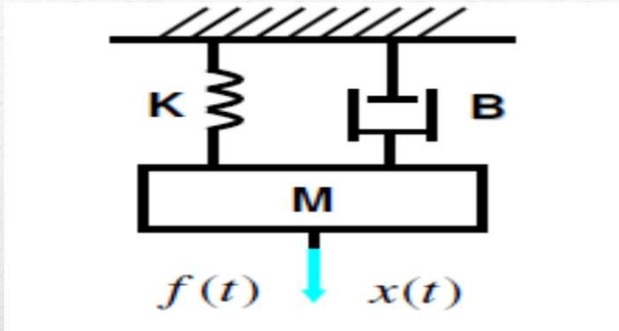
Note: $x(t)$ represents the displacement change for spring resting position

Using Newton's Law: $F = M\ddot{x}$

$$M\ddot{x}(t) = f(t) - f_K(t) - f_B(t) = f(t) - Kx(t) - B\dot{x}(t)$$

$$M\ddot{x}(t) + Kx(t) + B\dot{x}(t) = f(t)$$

Modeling – Spring-Mass-Damper System



- Equation of motion

$$M\ddot{x}(t) + B\dot{x}(t) + Kx(t) = f(t)$$

- By Laplace transform (with zero initial conditions),

$$X(s) = \frac{1}{Ms^2 + Bs + K} F(s)$$

(2nd order system)

Modeling – Spring-Mass-Damper System

Example 2.16

Transfer Function—One Equation of Motion

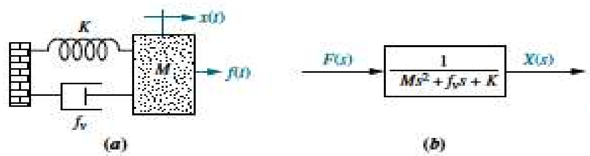


FIGURE 2.15 a. Mass, spring, and damper system; b. block diagram

PROBLEM: Find the transfer function, $X(s)/F(s)$, for the system of Figure 2.15(a).

SOLUTION: Begin the solution by drawing the free-body diagram shown in Figure 2.16(a). Place on the mass all forces felt by the mass. We assume the mass is traveling toward the right. Thus, only the applied force points to the right; all other forces impede the motion and act to oppose it. Hence, the spring, viscous damper, and the force due to acceleration point to the left.

We now write the differential equation of motion using Newton's law to sum to zero all of the forces shown on the mass in Figure 2.16(a):

$$M\frac{d^2x(t)}{dt^2}+f_v\frac{dx(t)}{dt}+Kx(t)=f(t) \qquad (2.108)$$

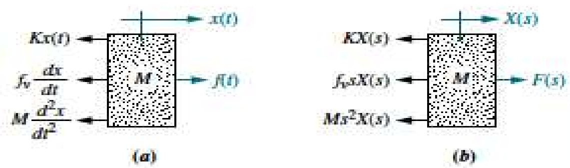


FIGURE 2.16 a. Free-body diagram of mass, spring, and damper system; b. transformed free-body diagram

Taking the Laplace transform, assuming zero initial conditions,

$$Ms^2X(s)+f_vsX(s)+KX(s)=F(s) \qquad (2.109)$$

or

$$(Ms^2+f_vs+K)X(s)=F(s) \qquad (2.110)$$

Solving for the transfer function yields

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{Ms^2+f_vs+K} \qquad (2.111)$$

which is represented in Figure 2.15(b).

Modeling – Spring-Mass-Damper System

Example 2.17

Transfer Function—Two Degrees of Freedom

PROBLEM: Find the transfer function, $X_2(s)/F(s)$, for the system of Figure 2.17(a).

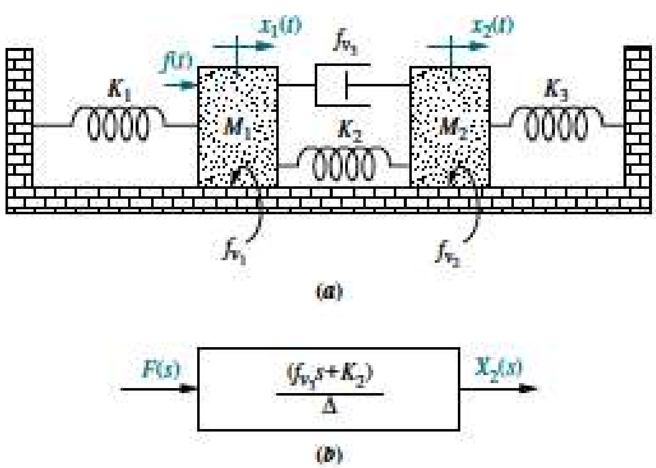


FIGURE 2.17 **a.** Two-degrees-of-freedom translational mechanical system;[8] **b.** block diagram

Modeling – Spring-Mass-Damper System

SOLUTION: The system has two degrees of freedom, since each mass can be moved in the horizontal direction while the other is held still. Thus, two simultaneous equations of motion will be required to describe the system. The two equations come from free-body diagrams of each mass. Superposition is used to draw the free-body diagrams. For example, the forces on M_1 are due to (1) its own motion and (2) the motion of M_2 transmitted to M_1 through the system. We will consider these two sources separately.

If we hold M_2 still and move M_1 to the right, we see the forces shown in Figure 2.18(a). If we hold M_1 still and move M_2 to the right, we see the forces shown in Figure 2.18(b). The total force on M_1 is the superposition, or sum, of the forces just discussed. This result is shown in Figure 2.18(c). For M_2, we proceed in a similar fashion: First we move M_2 to the right while holding M_1 still; then we move M_1 to the right and hold M_2 still. For each case we evaluate the forces on M_2. The results appear in Figure 2.19.

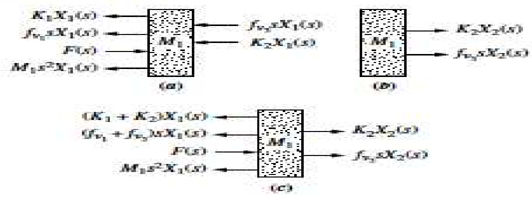


FIGURE 2.18 a. Forces on M_1 due only to motion of M_1; b. forces on M_1 due only to motion of M_2; c. all forces on M_1

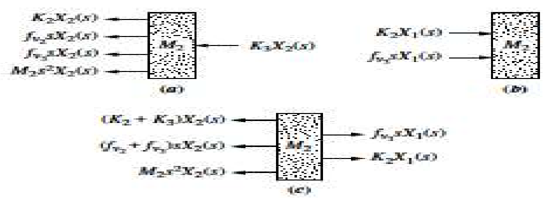


FIGURE 2.19 a. Forces on M_2 due only to motion of M_2; b. forces on M_2 due only to motion of M_1; c. all forces on M_2

The Laplace transform of the equations of motion can now be written from Figures 2.18(c) and 2.19(c) as

$$[M_1 s^2 (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s) \quad (2.118a)$$

$$-(f_{v_3}s + K_2)X_1(s) + [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0 \quad (2.118b)$$

From this, the transfer function, $X_2(s)/F(s)$, is

$$\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v_3}s + K_2)}{\Delta} \quad (2.119)$$

as shown in Figure 2.17(b) where

$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)] & -(f_{v_3}s + K_2) \\ -(f_{v_3}s + K_2) & [M_2 s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)] \end{vmatrix}$$

Modeling – Spring-Mass-Damper System

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_1\end{bmatrix} \qquad (2.120a)$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix} \qquad (2.120b)$$

Modeling – Spring-Mass-Damper System

Example 2.18

Equations of Motion by Inspection

PROBLEM: Write, but do not solve, the equations of motion for the mechanical network of Figure 2.20.

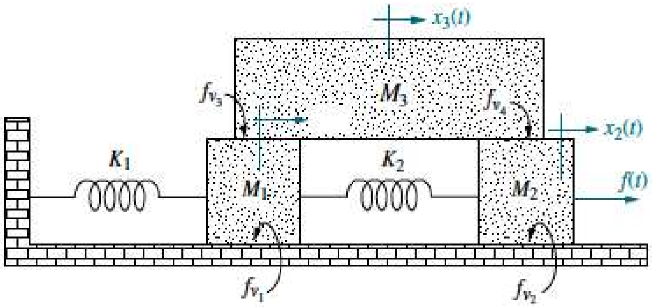


FIGURE 2.20 Three-degrees-of-freedom translational mechanical system

Modeling – Spring-Mass-Damper System

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_1\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_1\end{bmatrix}$$

Similarly, for M_2 and M_3, respectively,

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_2\end{bmatrix} X_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_2\end{bmatrix} X_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_2\end{bmatrix}$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_1 \text{ and } x_3\end{bmatrix} X_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ x_2 \text{ and } x_3\end{bmatrix} X_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } x_3\end{bmatrix} X_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied forces}\\ \text{at } x_3\end{bmatrix}$$

Modeling – Spring-Mass-Damper System

for M_1

$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - K_2X_2(s) - f_{v_3}sX_3(s) = 0$$

for M_2,

$$-K_2X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_4})s + K_2]X_2(s) - f_{v_4}sX_3(s) = F(s)$$

for M_3,

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + [M_3s^2 + (f_{v_3} + f_{v_4})s]X_3(s) = 0$$

Exercise

Skill-Assessment Exercise 2.8

PROBLEM: Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

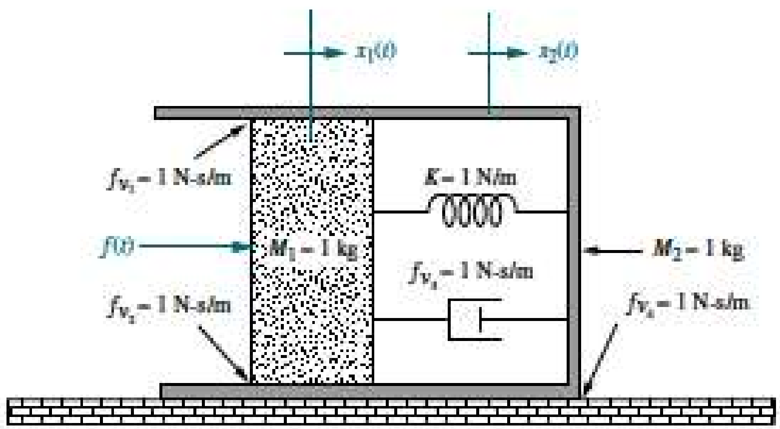


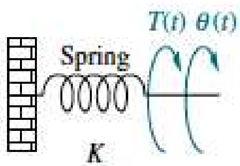
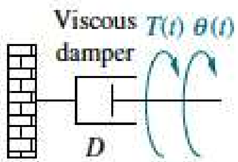
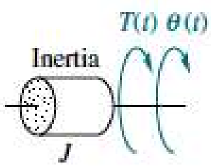
FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

ANSWER: $G(s) = \dfrac{3s+1}{s(s^3+7s^2+5s+1)}$

The complete solution is at www.wiley.com/college/nise.

Rotational Mechanical System Transfer Functions

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

Rotational Mechanical System Transfer Functions

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

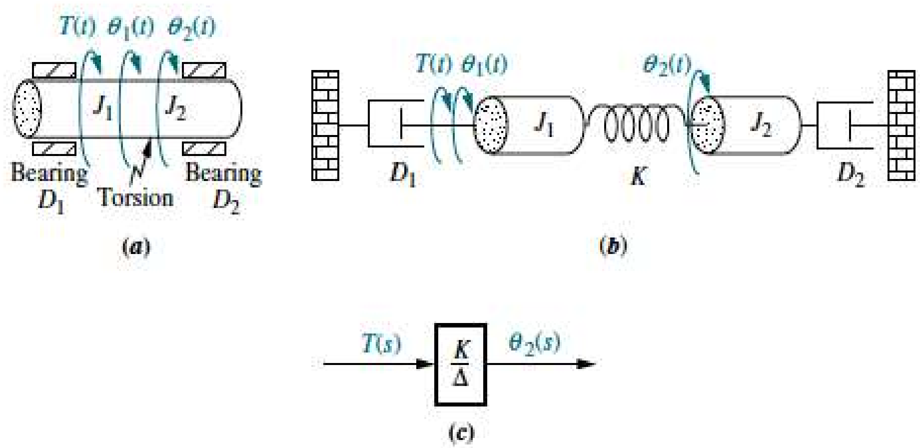


FIGURE 2.22 **a.** Physical system; **b.** schematic; **c.** block diagram

Rotational Mechanical System Transfer Functions

TryIt 2.9

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.128).

```
syms s J1 D1 K T J2 D2...
  theta1 theta2
A=[(J1*s^2+D1*s+K) -K
    -K (J2*s^2+D2*s+K)];
B=[theta1
    theta2];
C=[T
    0];
B=inv(A)*C;
theta2=B(2);
'theta2'
pretty(theta2)
```

$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s) \tag{2.127a}$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0 \tag{2.127b}$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \tag{2.128}$$

as shown in Figure 2.22(c), where

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

Notice that Eq. (2.127) have that now well-known form

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix} \tag{2.129a}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix} \tag{2.129b}$$

Rotational Mechanical System Transfer Functions

Example 2.20

Equations of Motion By Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

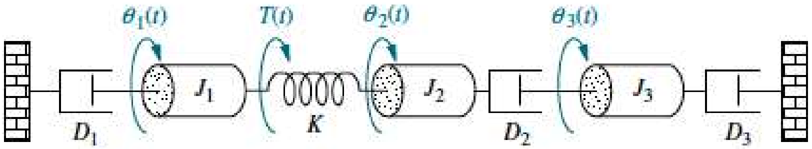


FIGURE 2.25 Three-degrees-of-freedom rotational system

Rotational Mechanical System Transfer Functions

SOLUTION: The equations will take on the following form, similar to electrical mesh equations:

$$\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_1\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_1\end{bmatrix} \quad (2.130\text{a})$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{bmatrix}\theta_1(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_2\end{bmatrix}\theta_2(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_2\end{bmatrix} \quad (2.130\text{b})$$

$$-\begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{bmatrix}\theta_1(s) - \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{bmatrix}\theta_2(s) + \begin{bmatrix}\text{Sum of}\\ \text{impedances}\\ \text{connected}\\ \text{to the motion}\\ \text{at } \theta_3\end{bmatrix}\theta_3(s) = \begin{bmatrix}\text{Sum of}\\ \text{applied torques}\\ \text{at } \theta_3\end{bmatrix} \quad (2.130\text{c})$$

Hence,

$$\begin{aligned}
(J_1s^2 + D_1s + K)\theta_1(s) \quad & -K\theta_2(s) & -0\theta_3(s) &= T(s)\\
-K\theta_1(s) \quad & +(J_2s^2 + D_2s + K)\theta_2(s) & -D_2s\theta_3(s) &= 0\\
-0\theta_1(s) \quad & -D_2s\theta_2(s) & +(J_3s^2 + D_3s + D_2s)\theta_3(s) &= 0
\end{aligned}$$

(2.131a, b, c)

Transfer functions for systems with gears

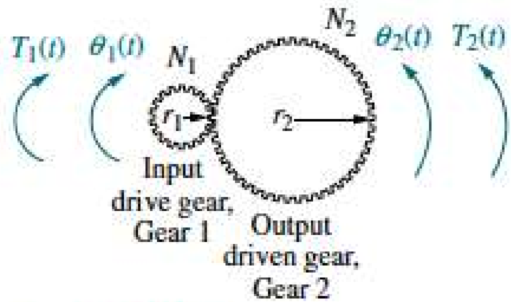


FIGURE 2.27 A gear system

From Figure 2.27, as the gears turn, the distance traveled along each gear's circumference is the same. Thus,

$$r_1\theta_1 = r_2\theta_2 \quad (2.132)$$

or

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} \quad (2.133)$$

Relationship between input torque and delivered torque

What is the relationship between the input torque, T_1, and the delivered torque, T_2? If we assume the gears are *lossless*, that is they do not absorb or store energy,

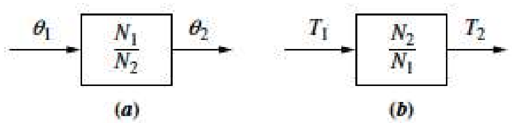


FIGURE 2.28 Transfer functions for **a.** angular displacement in lossless gears and **b.** torque in lossless gears

$$T_1\theta_1 = T_2\theta_2$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

EXAMPLE

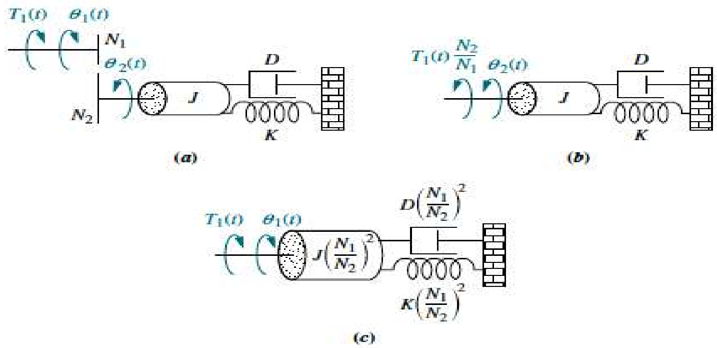


FIGURE 2.29 **a.** Rotational system driven by gears; **b.** equivalent system at the output after reflection of input torque; **c.** equivalent system at the input after reflection of impedances

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

$$\left[J\left(\frac{N_1}{N_2}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

Generalizing the results, we can make the following statement: *Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedance by the ratio*

$$\left(\frac{\text{Number of teeth of gear on } \textit{destination} \text{ shaft}}{\text{Number of teeth of gear on } \textit{source} \text{ shaft}}\right)^2$$

EXAMPLE

Example 2.21

Transfer Function—System with Lossless Gears

PROBLEM: Find the transfer function, $\theta_2(s)/T_1(s)$, for the system of Figure 2.30(a).

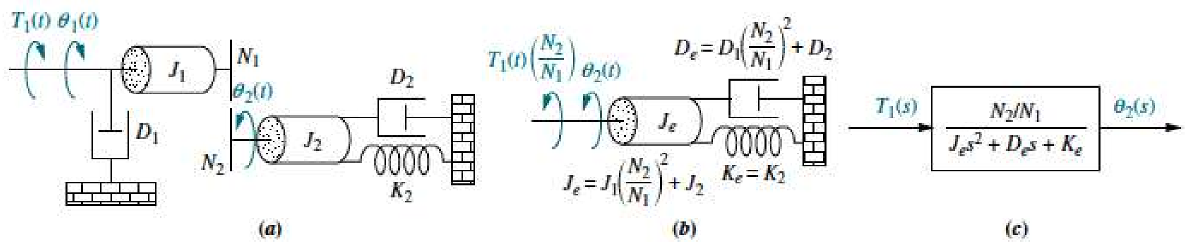


FIGURE 2.30 a. Rotational mechanical system with gears; b. system after reflection of torques and impedances to the output shaft; c. block diagram

SOLUTION:

Let us first reflect the impedances (J_1 and D_1) and torque (T_1) on the input shaft to the output as shown in Figure 2.30(b), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1). The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1} \tag{2.139}$$

where

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2; \quad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2; \quad K_e = K_2$$

Solving for $\theta_2(s)/T_1(s)$, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2/N_1}{J_e s^2 + D_e s + K_e} \tag{2.140}$$

as shown in Figure 2.30(c).

Gear train

In order to eliminate gears with large radii, a *gear train* is used to implement large gear ratios by cascading smaller gear ratios. A schematic diagram of a gear train is shown in Figure 2.31. Next to each rotation, the angular displacement relative to θ_1 has been calculated. From Figure 2.31,

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1 \qquad (2.141)$$

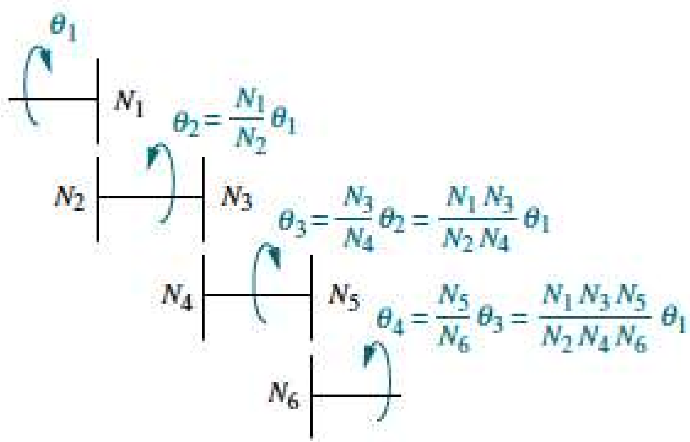


FIGURE 2.31 Gear train

Gear train

Example 2.22

Transfer Function—Gears with Loss

PROBLEM: Find the transfer function, $\theta_1(s)/T_1(s)$, for the system of Figure 2.32(a).

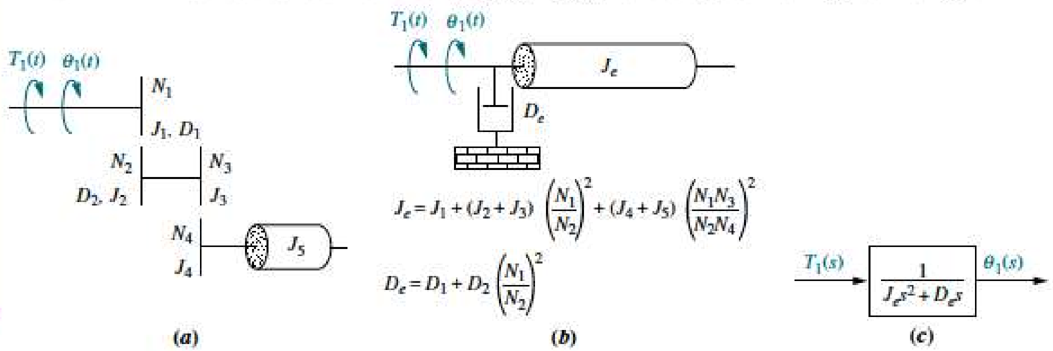


FIGURE 2.32
a. System using a gear train; **b.** equivalent system at the input; **c.** block diagram

$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

where

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$

and

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

The transfer function is

$$G(s) = \frac{\theta_1(s)}{T_1(s)} = \frac{1}{J_e s^2 + D_e s}$$

Electromechanical system transfer function

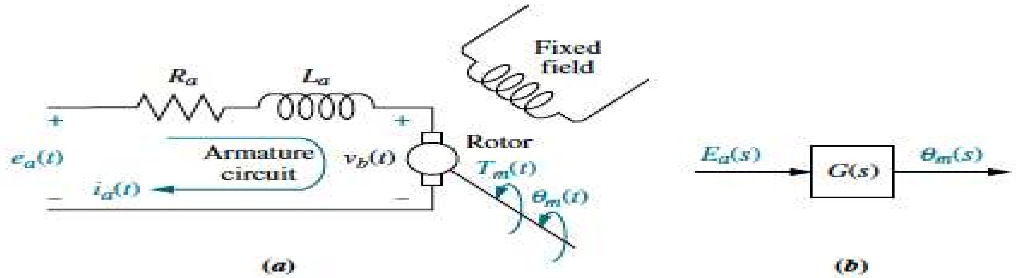


FIGURE 2.35 DC motor: a. schematic;[12] b. block diagram

In Figure 2.35(a) a magnetic field is developed by stationary permanent magnets or a stationary electromagnet called the *fixed field*. A rotating circuit called the *armature*, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $F = Bli_a(t)$, where B is the magnetic field strength and l is the length of the conductor. The resulting torque turns the *rotor*, the rotating member of the motor.

There is another phenomenon that occurs in the motor: A conductor moving at right angles to a magnetic field generates a voltage at the terminals of the conductor equal to $e = Blv$, where e is the voltage and v is the velocity of the conductor normal to the magnetic field. Since the current-carrying armature is rotating in a magnetic field, its voltage is proportional to speed. Thus,

$$v_b(t) = K_b \frac{d\theta_m(t)}{dt}$$

Electromechanical system transfer function

We call $v_b(t)$ the *back electromotive force (back emf)*; K_b is a constant of proportionality called the back emf constant; and $d\theta_m(t)/dt = \omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s\theta_m(s)$$

The relationship between the armature current, $i_a(t)$, the applied armature voltage, $e_a(t)$, and the back emf, $v_b(t)$, is found by writing a loop equation around the Laplace transformed armature circuit

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

The torque developed by the motor is proportional to the armature current; thus,

$$T_m(s) = K_t I_a(s)$$

where T_m is the torque developed by the motor, and K_t is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics. In a consistent set of units, the value of K_t is equal to the value of

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

Electromechanical system transfer function

FIGURE 2.36 Typical equivalent mechanical loading on a motor

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s)$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)}$$

Electromechanical system transfer function

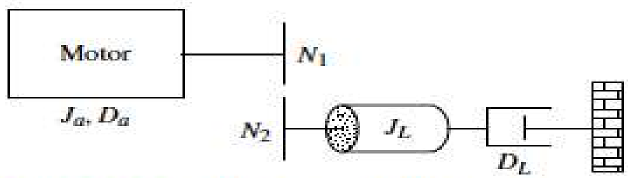


FIGURE 2.37 DC motor driving a rotational mechanical load

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2; \; D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2$$

with $L_a = 0$,

$$\frac{R_a}{K_t}T_m(s) + K_b s\theta_m(s) = E_a(s)$$

Taking the inverse Laplace transform, we get

$$\frac{R_a}{K_t}T_m(t) + K_b\omega_m(t) = e_a(t)$$

Electromechanical system transfer function

$$T_{\text{stall}} = \frac{K_t}{R_a} e_a$$

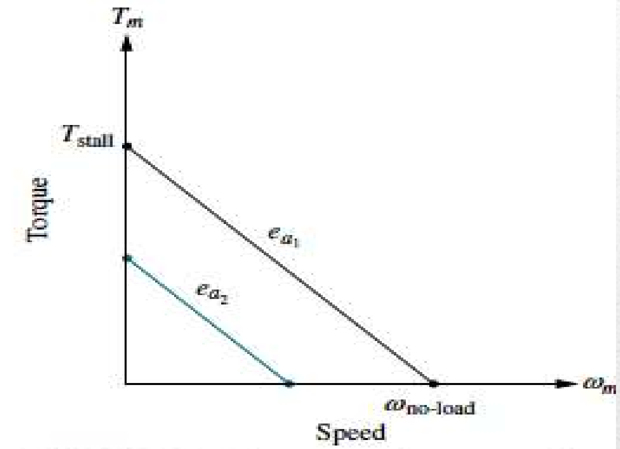


FIGURE 2.38 Torque-speed curves with an armature voltage, e_a, as a parameter

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a}$$

and

$$K_b = \frac{e_a}{\omega_{\text{no-load}}}$$

$$T_m = -\frac{K_b K_t}{R_a}\omega_m + \frac{K_t}{R_a} e_a$$

The torque axis intercept occurs when the angular velocity reaches zero. That value of torque is called the *stall torque*, T_{stall}. Thus,

Electromechanical system transfer function

Example 2.23

Transfer Function—DC Motor and Load

PROBLEM: Given the system and torque-speed curve of Figure 2.39(*a*) and (*b*), find the transfer function, $\theta_L(s)/E_a(s)$.

SOLUTION: Begin by finding the mechanical constants, J_m and D_m, in Eq. (2.153). From Eq. (2.155), the total inertia at the armature of the motor is

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \tag{2.164}$$

and the total damping at the armature of the motor is

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \tag{2.165}$$

Now we will find the electrical constants, K_t/R_a and K_b. From the torque-speed curve of Figure 2.39(*b*),

$$T_{stall} = 500 \tag{2.166}$$

$$\omega_{no\text{-}load} = 50 \tag{2.167}$$

$$e_a = 100 \tag{2.168}$$

Virtual Experiment 2.2
Open-Loop Servo Motor

Put theory into practice exploring the dynamics of the Quanser Rotary Servo System modeled in LabVIEW. It is particularly important to know how a servo motor behaves when using them in high-precision applications such as hard disk drives.

Virtual experiments are found on WileyPLUS.

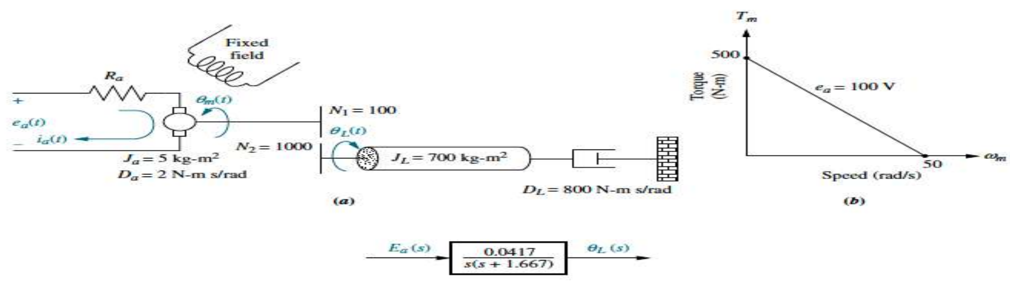


FIGURE 2.39 a. DC motor and load; b. torque-speed curve; c. block diagram

Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5 \tag{2.169}$$

and

$$K_b = \frac{e_a}{\omega_{no\text{-}load}} = \frac{100}{50} = 2 \tag{2.170}$$

Substituting Eqs. (2.164), (2.165), (2.169), and (2.170) into Eq. (2.153) yield

$$\frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s\left\{s + \frac{1}{12}[10 + (5)(2)]\right\}} = \frac{0.417}{s(s+1.667)} \tag{2.171}$$

In order to find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)} \tag{2.172}$$

as shown in Figure 2.39(*c*).

Electromechanical system transfer function

Skill-Assessment Exercise 2.11

WileyPLUS
WPCS
Control Solutions

PROBLEM: Find the transfer function, $G(s) = \theta_L(s)/E_a(s)$, for the motor and load shown in Figure 2.40. The torque-speed curve is given by $T_m = -8\omega_m + 200$ when the input voltage is 100 volts.

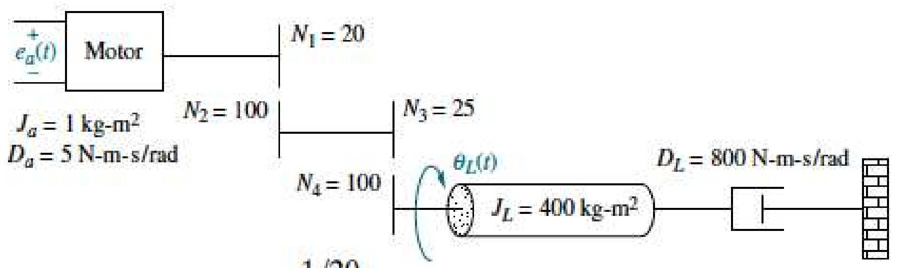


FIGURE 2.40 Electromechanical system for Skill-Assessment Exercise 2.11

ANSWER: $G(s) = \dfrac{1/20}{s[s + (15/2)]}$

The complete solution is at www.wiley.com/college/nise.

ELECTRIC CIRCUIT ANALOG

An electric circuit that is analogous to a system from another discipline is called an electric circuit *analog*. Analogs can be obtained by comparing the describing equations, such as the equations of motion of a mechanical system, with either electrical mesh or nodal equations. When compared with mesh equations, the resulting electrical circuit is called a *series analog*. When compared with nodal equations, the resulting electrical circuit is called a *parallel analog*.

Series Analog

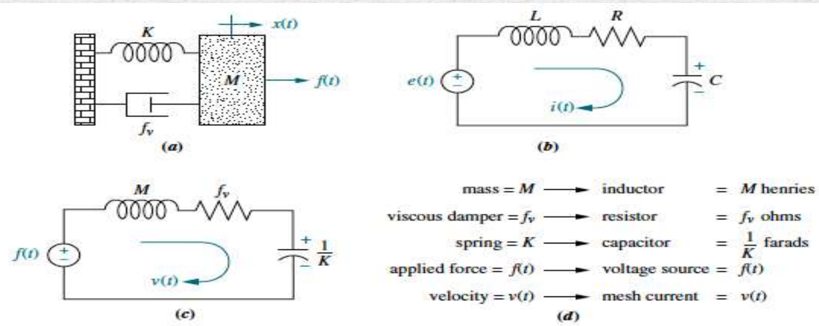


FIGURE 2.41 Development of series analog: **a.** mechanical system; **b.** desired electrical representation; **c.** series analog; **d.** parameters for series analog

Consider the translational mechanical system shown in Figure whose equation of motion is

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

Kirchhoff's mesh equation for the simple series RLC network shown in Figure is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

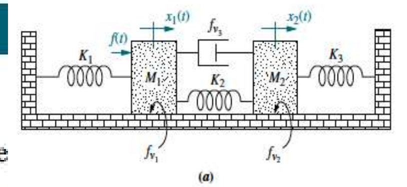
$$\frac{Ms^2 + f_v s + K}{s} sX(s) = \left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s)$$

Electric circuit analog

Example 2.24

Converting a Mechanical System to a Series Analog

PROBLEM: Draw a series analog for the mechanical system of Figure



SOLUTION: Equations (2.118) are analogous to electrical mesh equations after conversion to velocity. Thus,

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s) \qquad (2.176a)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0 \qquad (2.176b)$$

Coefficients represent sums of electrical impedance. Mechanical impedances associated with M_1 form the first mesh, where impedances between the two masses are common to the two loops. Impedances associated with M_2 form the second mesh. The result is shown in Figure 2.42, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2, respectively.

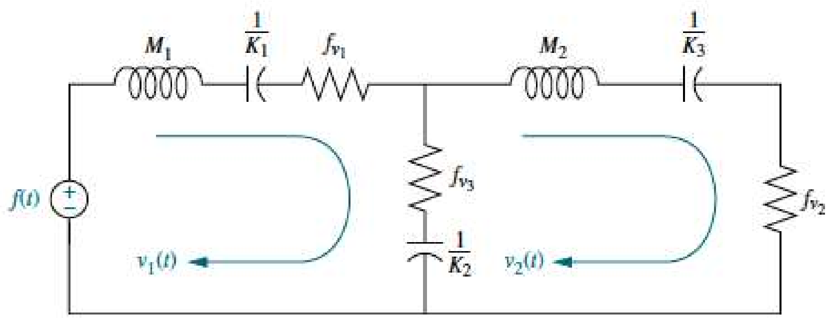


FIGURE 2.42 Series analog of mechanical system of Figure 2.17(*a*)

Electric circuit analog

Parallel Analog

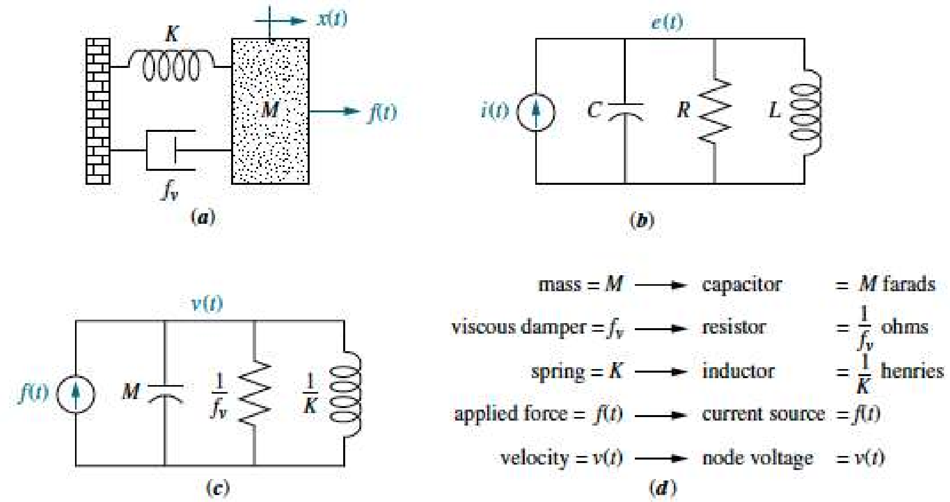


FIGURE 2.43 Development of parallel analog: **a.** mechanical system; **b.** desired electrical representation; **c.** parallel analog; **d.** parameters for parallel analog

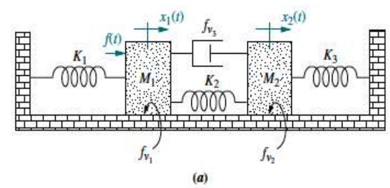
$$\left(Cs+\frac{1}{R}+\frac{1}{Ls}\right)E(s)=I(s)$$

Electric circuit analog

Example 2.25

Converting a Mechanical System to a Parallel Analog

PROBLEM: Draw a parallel analog for the mechanical system of Figur



SOLUTION: Equation (2.176) is also analogous to electrical node equations. Coefficients represent sums of electrical admittances. Admittances associated with M_1 form the elements connected to the first node, where mechanical admittances between the two masses are common to the two nodes. Mechanical admittances associated with M_2 form the elements connected to the second node. The result is shown in Figure 2.44, where $v_1(t)$ and $v_2(t)$ are the velocities of M_1 and M_2, respectively.

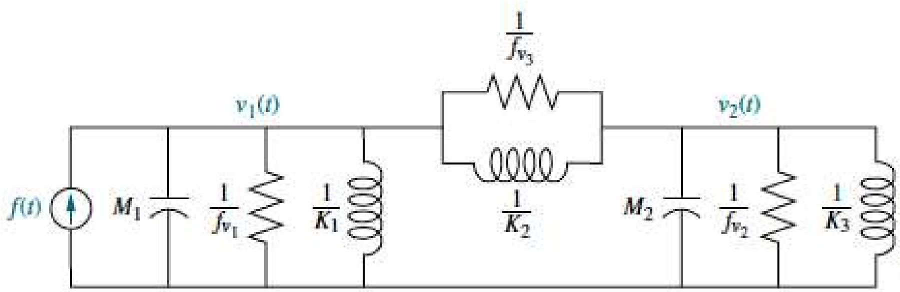


FIGURE 2.44 Parallel analog of mechanical system of Figure 2.17(*a*)